EXAMPLES FOR CROSS CURVATURE FLOW ON 3-MANIFOLDS

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ABSTRACT. Recently, B.Chow and R.S.Hamilton [3] introduced the cross curvature flow on 3-manifolds. In this paper, we analyze two interesting examples for this new flow. One is on a square torus bundle over a circle, and the other is on a S^2 bundle over a circle. We show that the global flow exist in both cases. But on the former the flow diverges at time infinity, and on the latter the flow converges at time infinity.

1. INTRODUCTION

Recently, B.Chow and R.S.Hamilton [3] introduced the cross curvature flow on 3-manifolds. This flow has a very nice property that it has at least two invariant sets of Riemannian metrics. In particular, the set of metrics of negative curvature is preserved. So one may try to use this flow to study the Geometrization Conjecture.

The aim of this paper is give two examples which show that the cross curvature flow is very degenerate in some cases. We show that the global flow may diverge at time infinity in a square torus bundle.

We now begin with the definition of the cross curvature flow.

Let (M, g) be a 3-dimensional Riemannian manifold. The Einstein tensor is

$$P_{ij} = R_{ij} - \frac{1}{2}Rg_{ij},$$

where R_{ij} is the Ricci tensor and R is the scalar curvature of the metric g. Let

(1) $$P^{ij} = g^{ik}g^{jl}R_{kl} - \frac{1}{2}Rg^{ij}.$$

Then B.Chow and R.Hamilton [3] define the cross curvature tensor by

$$h_{ij} = (\frac{\det P^{kl}}{\det g^{kl}}) \cdot V_{ij},$$

where V_{ij} is the inverse of P^{ij}. Clearly from this definition and the matrix theory we know that the cross curvature can also be defined even if P^{ij} has

no inverse. The cross curvature flow is defined by

$$(2) \qquad \frac{\partial}{\partial t} g_{ij} = 2h_{ij},$$

if the sectional curvature is negative or

$$(3) \qquad \frac{\partial}{\partial t} g_{ij} = -2h_{ij}$$

if the sectional curvature is positive. In some cases, B.Andrews [1] proved the convergence results.

We remark that the validity of definition of h_{ij} for P^{ij} being degenerate can also be seen from the following fact. Let μ_{ijk} be the volume form of g with the normalization that $\mu_{123} = 1$ and raise indices by

$$\mu^{ijk} = g^{ip}g^{jq}g^{kr}\mu_{pqr}.$$

Then the cross curvature can be written as

$$h_{ij} = \frac{1}{8}\mu^{pqk}\mu^{rsl}R_{ilpq}R_{kjrs}.$$

where R_{ijkl} is the Riemannian curvature tensor of g.

Roughly speaking, we show the following results:
(a) The cross curvature flow (XCF)(2) has global solutions on a square torus bundle over a circle and the global flow diverges at time infinity.
(b) The cross curvature flow (XCF)(3) has global solutions on a S^2 bundle over a circle and the global flow converges at time infinity.

For more precise assumptions on the initial metrics, one can consult Theorems 1 and 11,14 in sections 3 and 4. We point out that we reduce the XCF into degenerate parabolic equations. This is the difficulty which does not appear in the Ricci-Hamilton flow [6]. In section two, we recall some basic formulae in Riemannian Geometry.

2. FUNDAMENTAL LOCAL FORMULAS

It is convenient to compute using moving frame method. So, first of all, let us review some useful formulae.

Let M be a Riemannian manifold of dimension n. $\omega_i, i = 1, \cdots, n$, is an orthogonal co-frame on M. e_i is its dual frame. Then we have the following structure equations:

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji},$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}.$$

The forms ω_{ij}, which are anti-symmetric in i, j, are the connection forms. The Ω_{ij}, also anti-symmetric in i, j, are the curvature forms. They have the expression

$$(4) \qquad \Omega_{ij} = -\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

where the coefficients R_{ijkl} satisfy the well-known symmetry relations

$$R_{ijkl} = -R_{jikl} = -R_{ijlk},$$
$$R_{ijkl} = R_{klij},$$
$$R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

The Ricci tensor R_{ij} is defined by

(5) $$R_{ij} = \sum_k R_{ikjk},$$

and the scalar curvature by

(6) $$R = \sum_i R_{ii}.$$

In follows, we take $n = 3$.

3. XCF on a square torus bundle over a circle

Consider a 3-manifold M^3 where the torus group $T^2 = S^1 \times S^1$ acts freely. Then M^3 is a T^2 bundle over the circle S^1. There is a larger group G which is the isometry group of the square flat torus R^2/Z^2, containing T^2 as a subgroup. Consider metrics on M^3 which have G as their isometry group with the subgroup T^2 acting freely. These 3-manifolds are called square torus bundles over a circle. For details, see [6]. We now consider the metric on a square torus bundle taking the form

$$ds^2 = f(x)^2 dx^2 + g(x)^2 (dy^2 + dz^2),$$

where x is a coordinate on the orbit space S^1 and, y and z are coordinates on the fibre, f, g are positive smooth functions on S^1. Note that $ds = f(x)dx$ is the arc length for the quotient metric on the orbit space S^1, and $g(x)$ is the length of the side of the square fibre over x. For this metric, we can take the orthogonal co-frame ω_i as

$$\omega_1 = f dx, \omega_2 = g dy, \omega_3 = g dz.$$

Then its dual frame is

$$e_1 = \frac{1}{f}\frac{\partial}{\partial x}, e_2 = \frac{1}{g}\frac{\partial}{\partial y}, e_3 = \frac{1}{g}\frac{\partial}{\partial z}.$$

A straightforward computation shows that the connection forms are

$$\omega_{12} = \frac{g_x}{f} dy, \omega_{13} = \frac{g_x}{f} dz, \omega_{23} = 0,$$

and the curvature forms are

$$\Omega_{12} = (\frac{g_x}{f})_x dx \wedge dy, \Omega_{13} = (\frac{g_x}{f})_x dx \wedge dz, \Omega_{23} = (\frac{g_x}{f})^2 dy \wedge dz.$$

Using relation (4), we get

$$R_{1212} = R_{1313} = -\frac{1}{fg}(\frac{g_x}{f})_x, R_{2323} = -\frac{1}{g^2}(\frac{g_x}{f})^2.$$

Other components of R_{ijkl} are zeros.

Substituting these expressions into equation (5), we get the Ricci tensor:

$$R_{11} = -\frac{2}{fg}(\frac{g_x}{f})_x, R_{22} = R_{33} = -\frac{1}{fg}(\frac{g_x}{f})_x - \frac{1}{g^2}(\frac{g_x}{f})^2.$$

Substituting these expression into equation (6), we get the scalar curvature:

$$R = -\frac{4}{fg}(\frac{g_x}{f})_x - \frac{2}{g^2}(\frac{g_x}{f})^2.$$

Now by equation (1), we have

$$P^{11} = \frac{1}{g^2}(\frac{g_x}{f})^2, P^{22} = P^{33} = \frac{1}{fg}(\frac{g_x}{f})_x.$$

Then we can compute the cross curvature tensor:

$$h_{11} = [\frac{1}{fg}(\frac{g_x}{f})_x]^2, h_{22} = h_{33} = \frac{1}{fg^3}(\frac{g_x}{f})^2(\frac{g_x}{f})_x.$$

For later use, we should express the cross curvature tensor in natural coordinates (x, y, z):

$$h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = f^2 h_{11} = [\frac{1}{g}(\frac{g_x}{f})_x]^2,$$

$$h(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = g^2 h_{22} = \frac{1}{fg}(\frac{g_x}{f})^2(\frac{g_x}{f})_x,$$

$$h(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = g^2 h_{33} = \frac{1}{fg}(\frac{g_x}{f})^2(\frac{g_x}{f})_x.$$

So the XCF (2)

$$\frac{\partial}{\partial t}g_{ij} = 2h_{ij}$$

reduces to the systems of degenerate evolution equations

$$\{ \begin{matrix} \frac{\partial f}{\partial t} = \frac{1}{fg^2}[(\frac{g_x}{f})_x]^2, \\ \frac{\partial g}{\partial t} = \frac{1}{fg^2}(\frac{g_x}{f})^2(\frac{g_x}{f})_x. \end{matrix}$$

Introduce the unit vector field on the orbit space

$$\frac{\partial}{\partial s} = \frac{1}{f}\frac{\partial}{\partial x}$$

whose evolution is given by the commutator

$$[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = -\frac{1}{f}\frac{\partial f}{\partial t}\frac{\partial}{\partial s}.$$

Then the XCF takes the form of the fully nonlinear degenerate parabolic equation

$$\frac{\partial g}{\partial t} = \frac{1}{g^2}(\frac{\partial g}{\partial s})^2\frac{\partial^2 g}{\partial s^2}$$

on a circle whose unit vector field $\frac{\partial}{\partial s}$ varies by the commutator

$$[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = -\frac{1}{g^2}(\frac{\partial^2 g}{\partial s^2})^2\frac{\partial}{\partial s}.$$

The local existence of the flow can be obtained from an approximate method. In fact, for $\epsilon > 0$, we consider

$$\frac{\partial g}{\partial t} = \frac{1}{g^2}[(\frac{\partial g}{\partial s})^2 + \epsilon]\frac{\partial^2 g}{\partial s^2}$$

We know from the standard parabolic equation theory that there is a local solution for every $\epsilon > 0$. Using the argument below (see Lemmata 2,3,and 4) we can show that we have a global solution for every $\epsilon > 0$ with uniform bounds of derivatives. Hence, by sending $\epsilon \to 0$, we can prove that the cross curvature flow (2) has a global solution for any initial data. Hence we have

Theorem 1. *The XCF on a square torus bundle over a circle has a solution which exists for all time.*

Let

$$g_{max}(t) = \max_{\theta \in S^1} g(\theta, t),$$

and

$$g_{min}(t) = \min_{\theta \in S^1} g(\theta, t).$$

Throughout this section, we will assume that

$$g_{max}(0) > g_{min}(0).$$

Lemma 2. *For all $t > 0$, $g_{max}(t) = g_{max}(0)$ and $g_{min}(t) = g_{min}(0)$.*

Proof. At the local maximum of g,

$$\frac{\partial g}{\partial s} = 0,$$

so

$$\frac{\partial g}{\partial t} = 0,$$

for $t > 0$. The similar argument can be done for the local minimum of g. $\square$

We give here a remark. Using the commutator relation we have

$$\frac{\partial}{\partial t}\frac{\partial g}{\partial s} = \frac{1}{g^2}(\frac{\partial g}{\partial s})^2\frac{\partial^2}{\partial s^2}\frac{\partial g}{\partial s} + \frac{1}{g^2}\frac{\partial g}{\partial s}(\frac{\partial}{\partial s}\frac{\partial g}{\partial s})^2 - \frac{2}{g^3}(\frac{\partial g}{\partial s})^3\frac{\partial}{\partial s}\frac{\partial g}{\partial s}.$$

Assume that the zero points of g_s at $t = 0$ consist of maximum or minimum points of g on S^1. We know from the equations above and the maximum principle (see Proposition 1.2 in [2]) that the number of points with $g_s = 0$ is invariant along the flow.

Lemma 3. *There exists a constant $C < \infty$ such that for all $t > 0$,*

$$|\frac{\partial g}{\partial s}| \leq C.$$

Proof. Recall that

$$\frac{\partial}{\partial t}\frac{\partial g}{\partial s} = \frac{1}{g^2}(\frac{\partial g}{\partial s})^2\frac{\partial^2}{\partial s^2}\frac{\partial g}{\partial s} + \frac{1}{g^2}\frac{\partial g}{\partial s}(\frac{\partial}{\partial s}\frac{\partial g}{\partial s})^2 - \frac{2}{g^3}(\frac{\partial g}{\partial s})^3\frac{\partial}{\partial s}\frac{\partial g}{\partial s}.$$

This shows by the maximum principle that the maximum of $\frac{\partial g}{\partial s}$ decreases and the minimum increases. □

Lemma 4. *There exists a constant $C < \infty$ such that for all $t > 0$,*

$$|\frac{\partial^2 g}{\partial s^2}| \leq C\exp(Ct).$$

Proof. Repeatedly using the commutator relation we get

$$\frac{\partial}{\partial t}(\frac{\partial^2 g}{\partial s^2})^2 = \frac{1}{g^2}(\frac{\partial g}{\partial s})^2\frac{\partial^2}{\partial s^2}(\frac{\partial^2 g}{\partial s^2})^2 - \frac{2}{g^2}(\frac{\partial g}{\partial s})^2(\frac{\partial^3 g}{\partial s^3})^2 - \frac{4}{g^3}(\frac{\partial g}{\partial s})^3\frac{\partial}{\partial s}(\frac{\partial^2 g}{\partial s^2})^2$$

$$+\frac{4}{g^2}\frac{\partial g}{\partial s}\frac{\partial^2 g}{\partial s^2}\frac{\partial}{\partial s}(\frac{\partial^2 g}{\partial s^2})^2 + [\frac{12}{g^4}(\frac{\partial g}{\partial s})^4 - 16\frac{\partial}{\partial t}\log(g)](\frac{\partial^2 g}{\partial s^2})^2.$$

At the maximum of $(\frac{\partial^2 g}{\partial s^2})^2$,

$$\frac{\partial}{\partial s}(\frac{\partial^2 g}{\partial s^2})^2 = 0,$$

i.e.,

$$\frac{\partial^2 g}{\partial s^2}\frac{\partial^3 g}{\partial s^3} = 0.$$

Then,

$$\frac{\partial^3 g}{\partial s^3} = 0.$$

Otherwise, g will be constant which contracts Lemma 2.

By Lemmas 2 and 3, we have that at the maximum of $(\frac{\partial^2 g}{\partial s^2})^2$,

$$\frac{\partial}{\partial t}\log(g^{16}(\frac{\partial^2 g}{\partial s^2})^2) \leq C$$

for some constant $C < \infty$, from which we obtain the desired estimate. □

Now we make some geometrical observations about the XCF.

Lemma 5. *The length L of the orbit circle always increases.*

Proof. The arc length ds on the orbit circle varies by

$$\frac{\partial}{\partial t}ds = \frac{1}{g^2}(\frac{\partial^2 g}{\partial s^2})^2 ds$$

and the length

$$L = \int 1 ds$$

varies by

$$\frac{dL}{dt} = \int \frac{1}{g^2}(\frac{\partial^2 g}{\partial s^2})^2 ds \geq 0.$$

□

Theorem 6. *The length L tends to infinity as $t \to \infty$.*

Proof. If not, then the time derivative of the length L must be approaching zero at an ϵ-dense set of sufficiently large times. If we can gain some control of its time derivative, then we can conclude that it converges to zero.

$$\frac{d}{dt} \int \frac{1}{g^2} (\frac{\partial^2 g}{\partial s^2})^2 ds$$

$$= \int -\frac{38}{3} \frac{1}{g^5} (\frac{\partial g}{\partial s})^2 (\frac{\partial^2 g}{\partial s^2})^3 - \frac{1}{3} \frac{1}{g^4} (\frac{\partial^2 g}{\partial s^2})^4 - \frac{2}{g^4} (\frac{\partial g}{\partial s})^2 (\frac{\partial^3 g}{\partial s^3})^2$$

$$+ \frac{12}{g^6} (\frac{\partial g}{\partial s})^4 (\frac{\partial^2 g}{\partial s^2})^2$$

$$\leq \int (\frac{1}{\epsilon} + \frac{12}{g^4} (\frac{\partial g}{\partial s})^4) \frac{1}{g^2} (\frac{\partial^2 g}{\partial s^2})^2 + \int (\epsilon \frac{361}{4} \frac{1}{g^4} (\frac{\partial g}{\partial s})^4 - \frac{1}{3}) \frac{1}{g^4} (\frac{\partial^2 g}{\partial s^2})^4.$$

We can choose ϵ small enough to make the integrand of the second term is negative. Then we have

$$\frac{d}{dt} \int \frac{1}{g^2} (\frac{\partial^2 g}{\partial s^2})^2 ds \leq C \int \frac{1}{g^2} (\frac{\partial^2 g}{\partial s^2})^2 ds$$

for some constant $C < \infty$. This bounds the growth of the integral to exponential. Hence it converges to zero as $t \to \infty$. This leads to

$$\lim_{t \to \infty} \int (\frac{\partial^2 g}{\partial s^2})^2 ds = 0.$$

Then an easy argument shows that g converges to a constant. This contracts Lemma 2. $\qquad \square$

Corollary 7. *The total volume V of the bundle increases to infinity.*

Proof. Since

$$V = \int g^2 ds$$

we compute

$$\frac{dV}{dt} = \int \frac{2}{3} \frac{1}{g^2} (\frac{\partial g}{\partial s})^4 + (\frac{\partial^2 g}{\partial s^2})^2 \geq 0.$$

Furthermore,

$$V \geq g_{min}(0)^2 L.$$

Then Lemma 2 and Theorem 6 show that $V \to \infty$ as $t \to \infty$. $\qquad \square$

4. XCF ON A S^2 BUNDLE OVER A CIRCLE

Consider a 3-manifold M^3 where S^2 acts freely. Then M^3 is a S^2 bundle over the circle S^1. Topologically M^3 is $S^2 \times S^1$. In this section, we consider the metric on $S^2 \times S^1$ which takes the form

$$ds^2 = f(x)^2 dx + g(x)^2 (dy^2 + \cos^2 y dz^2).$$

Geometrically, $ds = f(x)dx$ is the arc length for the quotient metric on the orbit space S^1, and $g(x)$ is the length of the radius of the S^2 fibre over x. For this metric, we can take the orthogonal co-frame ω_i as

$$\omega_1 = f dx, \omega_2 = g dy, \omega_3 = g \cos y dz.$$

Then its dual frame is

$$e_1 = \frac{1}{f}\frac{\partial}{\partial x}, e_2 = \frac{1}{g}\frac{\partial}{\partial y}, e_3 = \frac{1}{g\cos y}\frac{\partial}{\partial z}.$$

The connection forms are

$$\omega_{12} = \frac{g_x}{f} dy, \omega_{13} = \frac{g_x}{f}\cos y dz, \omega_{23} = -\sin y dz.$$

The curvature forms are

$$\Omega_{12} = (\frac{g_x}{f})_x dx \wedge dy, \Omega_{13} = (\frac{g_x}{f})_x \cos y dx \wedge dz,$$

$$\Omega_{23} = [(\frac{g_x}{f})^2 - 1] \cos y dy \wedge dz.$$

Using equation (4), we get

$$R_{1212} = R_{1313} = -\frac{1}{fg}(\frac{g_x}{f})_x, R_{2323} = -\frac{1}{g^2}[(\frac{g_x}{f})^2 - 1].$$

Other components are zeros.

Then the Ricci tensor is

$$R_{11} = -\frac{2}{fg}(\frac{g_x}{f})_x, R_{22} = R_{33} = -\frac{1}{fg}(\frac{g_x}{f})_x - \frac{1}{g^2}[(\frac{g_x}{f})^2 - 1].$$

The scalar curvature is

$$R = -\frac{4}{fg}(\frac{g_x}{f})_x - \frac{2}{g^2}[(\frac{g_x}{f})^2 - 1].$$

Now we can compute

$$P^{11} = \frac{1}{g^2}[(\frac{g_x}{f})^2 - 1], P^{22} = P^{33} = \frac{1}{fg}(\frac{g_x}{f})_x.$$

This leads to

$$h_{11} = [\frac{1}{fg}(\frac{g_x}{f})_x]^2, h_{22} = h_{33} = \frac{1}{fg^3}[(\frac{g_x}{f})^2 - 1](\frac{g_x}{f})_x.$$

Still, we need the expression of the cross curvature tensor in natural coordinates:

$$h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = f^2 h_{11} = [\frac{1}{g}(\frac{g_x}{f})_x]^2,$$

$$h(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = g^2 h_{22} = \frac{1}{fg}[(\frac{g_x}{f})^2 - 1](\frac{g_x}{f})_x,$$

$$h(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = g^2 \cos^2 y h_{33} = \frac{\cos^2 y}{fg}[(\frac{g_x}{f})^2 - 1](\frac{g_x}{f})_x$$

So the XCF

$$\frac{\partial}{\partial t} g_{ij} = -2h_{ij}$$

reduces to the systems of evolution equations

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial t} = -\frac{1}{fg^2}[(\frac{g_x}{f})_x]^2, \\ \frac{\partial g}{\partial t} = -\frac{1}{fg^2}[(\frac{g_x}{f})^2 - 1](\frac{g_x}{f})_x. \end{array} \right.$$

Define the unit vector field on the orbit space

$$\frac{\partial}{\partial s} = \frac{1}{f}\frac{\partial}{\partial x}$$

whose evolution is given by the commutator

$$[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = -\frac{1}{f}\frac{\partial f}{\partial t}\frac{\partial}{\partial s}.$$

Then the XCF takes the form of the fully nonlinear equation

$$\frac{\partial g}{\partial t} = \frac{1}{g^2}[1 - (\frac{\partial g}{\partial s})^2]\frac{\partial^2 g}{\partial s^2}$$

on a circle whose unit vector field $\frac{\partial}{\partial s}$ varies by the commutator

$$[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = \frac{1}{g^2}(\frac{\partial^2 g}{\partial s^2})^2\frac{\partial}{\partial s}.$$

Throughout this section, we will assume that the following initial condition holds

$$\max_{S^1}|\frac{\partial g}{\partial s}(\cdot, 0)| \le \frac{1}{4}.$$

Then the local solution exists by the standard parabolic equation theory.

As before, applying the maximum principle to the evolution of g, we directly have

Lemma 8. $g_{max}(t)$ *decreases and* $g_{min}(t)$ *increases.*

Lemma 9. *For all* $t > 0$, *we have*

$$\max_{S^1}|\frac{\partial g}{\partial s}(\cdot, t)| \le \frac{1}{4}.$$

Proof. Using the commutator relation

$$\frac{\partial}{\partial t}(\frac{\partial g}{\partial s})^2 = \frac{1}{g^2}(1 - (\frac{\partial g}{\partial s})^2)\frac{\partial^2}{\partial s^2}(\frac{\partial g}{\partial s})^2 - \frac{2}{g^3}\frac{\partial g}{\partial s}(1 - (\frac{\partial g}{\partial s})^2)\frac{\partial}{\partial s}(\frac{\partial g}{\partial s})^2$$

$$-\frac{2}{g^2}(\frac{\partial^2 g}{\partial s^2})^2.$$

The initial condition shows that

$$\frac{\partial}{\partial t}(\frac{\partial g}{\partial s})^2 \leq 0$$

at the maximum of $(\frac{\partial g}{\partial s})^2$. So the maximum decreases. $\qquad\square$

Lemma 10. *There exists a constant $C < \infty$ such that*

$$|\frac{\partial^2 g}{\partial s^2}| \leq C \exp(Ct)$$

for all $t \geq 0$.

Proof. Repeatedly using the commutator relation

$$(7) \quad \frac{\partial}{\partial t}\frac{\partial^2 g}{\partial s^2} = \frac{1}{g^2}(1 - (\frac{\partial g}{\partial s})^2)\frac{\partial^2}{\partial s^2}\frac{\partial^2 g}{\partial s^2} + (\frac{4}{g^3}(\frac{\partial g}{\partial s})^3 - \frac{4}{g^3}\frac{\partial g}{\partial s} - \frac{4}{g^2}\frac{\partial g}{\partial s}\frac{\partial^2 g}{\partial s^2})\frac{\partial^3 g}{\partial s^3}$$

$$+ \frac{6}{g^4}(\frac{\partial g}{\partial s})^2(1 - (\frac{\partial g}{\partial s})^2)\frac{\partial^2 g}{\partial s^2} - \frac{8}{g^3}(\frac{1}{4} - (\frac{\partial g}{\partial s})^2)(\frac{\partial^2 g}{\partial s^2})^2.$$

If g is constant, then the flow is over. So without loss of generality, we may assume g is not constant. Then at the maximum of $\frac{\partial^2 g}{\partial s^2}$,

$$\frac{\partial^2 g}{\partial s^2} > 0, \frac{\partial^3 g}{\partial s^3} = 0, \frac{\partial^4 g}{\partial s^4} \leq 0.$$

Note that the fourth term on the r.h.s. of equation (7) is non-positive. So

$$\frac{\partial}{\partial t}\frac{\partial^2 g}{\partial s^2} \leq C\frac{\partial^2 g}{\partial s^2}$$

for some $C > 0$. Therefore

$$(8) \qquad\qquad \frac{\partial^2 g}{\partial s^2} \leq C \exp(Ct)$$

for some $C > 0$.

Likewise, at the minimum of $\frac{\partial^2 g}{\partial s^2}$,

$$\frac{\partial^2 g}{\partial s^2} < 0, \frac{\partial^3 g}{\partial s^3} = 0, \frac{\partial^4 g}{\partial s^4} \geq 0.$$

Then

$$\frac{\partial}{\partial t}\log(-\frac{\partial^2 g}{\partial s^2}) \leq C - \frac{2}{g^3}(1 - (\frac{\partial g}{\partial s})^2)\frac{\partial^2 g}{\partial s^2} + \frac{6}{g^3}(\frac{\partial g}{\partial s})^2\frac{\partial^2 g}{\partial s^2}$$

$$\leq C - \frac{\partial}{\partial t}\log g^2$$

$$\Rightarrow$$

$$\frac{\partial}{\partial t}\log(-g^2\frac{\partial^2 g}{\partial s^2}) \leq C$$

$$\Rightarrow$$

$$(9) \qquad\qquad \frac{\partial^2 g}{\partial s^2} \geq -C \exp(Ct)$$

for some $C > 0$.

Combining (8) and (9), we obtain the desired estimate. $\qquad\square$

As a direct consequence, we have

Theorem 11. *Assume*

$$\max_{S^1} |\frac{\partial g}{\partial s}(\cdot, 0)| \le \frac{1}{4}.$$

The XCF on the S^2 bundle over a circle has a solution which exists for all time.

As before, we will give some geometrical results.

Lemma 12. *The length L of the orbit circle always decreases.*

Proof. The arc length ds on the orbit circle varies by

$$\frac{\partial}{\partial t} ds = -\frac{1}{g^2}(\frac{\partial^2 g}{\partial s^2})^2 ds.$$

Then the length varies by

$$\frac{dL}{dt} = -\int \frac{1}{g^2}(\frac{\partial^2 g}{\partial s^2})^2 ds \le 0.$$

$\square$

Lemma 13. *The L^2 norm of $\frac{\partial^2 g}{\partial s^2}$ converges to zero as $t \to \infty$.*

Proof. Since we have known that L decreases, so as in the proof of Theorem 6, what we need to do is to control

$$\frac{d}{dt} \int \frac{1}{g^2}(\frac{\partial^2 g}{\partial s^2})^2 ds.$$

A straightforward computation shows that

$$\frac{d}{dt} \int \frac{1}{g^2}(\frac{\partial^2 g}{\partial s^2})^2 ds$$

$$= \int -\frac{2}{g^4}(1 - (\frac{\partial g}{\partial s})^2)(\frac{\partial^3 g}{\partial s^3})^2 + \frac{1}{3g^4}(\frac{\partial^2 g}{\partial s^2})^4 + \frac{1}{g^5}(\frac{38}{3}(\frac{\partial g}{\partial s})^2 - 6)(\frac{\partial^2 g}{\partial s^2})^3$$

$$+ \frac{12}{g^6}(\frac{\partial g}{\partial s})^2(1 - (\frac{\partial g}{\partial s})^2)(\frac{\partial^2 g}{\partial s^2})^2.$$

Note that

$$-\frac{2}{g^4}(1 - (\frac{\partial g}{\partial s})^2) \ge -C_1$$

for some $C_1 > 0$.

Repeatedly using the *Hölder* inequality and the Peter-Paul inequality yields

$$\frac{d}{dt} \int \frac{1}{g^2}(\frac{\partial^2 g}{\partial s^2})^2 ds \le -C_1 \int (\frac{\partial^3 g}{\partial s^3})^2 ds + C_2(\sup(\frac{\partial^2 g}{\partial s^2})^2 + 1) \int (\frac{\partial^2 g}{\partial s^2})^2 ds$$

for some $C_2 > 0$.

But

$$\sup(\frac{\partial^2 g}{\partial s^2})^2 \le \int |\frac{\partial}{\partial s}(\frac{\partial^2 g}{\partial s^2})^2| ds$$

$$= 2 \int |\frac{\partial^2 g}{\partial s^2}| \cdot |\frac{\partial^3 g}{\partial s^3}| ds$$

$$\leq \int (\frac{\partial^2 g}{\partial s^2})^2 ds + \int (\frac{\partial^3 g}{\partial s^3})^2 ds.$$

Therefore

$$\frac{d}{dt} \int \frac{1}{g^2} (\frac{\partial^2 g}{\partial s^2})^2 ds$$

$$\leq (C_1 + C_2) \int (\frac{\partial^2 g}{\partial s^2})^2 ds + \sup(\frac{\partial^2 g}{\partial s^2})^2 \cdot (C_2 \int (\frac{\partial^2 g}{\partial s^2})^2 ds - C_1)$$

$$\leq C \int \frac{1}{g^2} (\frac{\partial^2 g}{\partial s^2})^2 ds + \sup(\frac{\partial^2 g}{\partial s^2})^2 \cdot (C_0 \int (\frac{\partial^2 g}{\partial s^2})^2 ds - C_1)$$

for some positive constant C and C_0. Here we have used Lemma 8. This bounds the growth of the integral to exponential when it is sufficiently small. Hence it converges to zero as $t \to \infty$. □

Theorem 14. *Assume*

$$\max_{S^1} |\frac{\partial g}{\partial s}(\cdot, 0)| \leq \frac{1}{4}.$$

Along the XCF on a S^2 bundle over a circle, g converges as $t \to \infty$ to a constant $\alpha > 0$.

Proof. Since

$$\sup(\frac{\partial g}{\partial s})^2 \leq \int |\frac{\partial}{\partial s}(\frac{\partial g}{\partial s})^2| ds$$

$$= 2 \int |\frac{\partial g}{\partial s}| \cdot |\frac{\partial^2 g}{\partial s^2}| ds$$

$$\leq C \cdot L^{\frac{1}{2}} \cdot (\int |\frac{\partial^2 g}{\partial s^2}|^2 ds)^{\frac{1}{2}}$$

$$\to 0$$

as $t \to \infty$, so

$$g_{max} - g_{min} \leq \int |\frac{\partial g}{\partial s}| ds$$

$$\leq \sup |\frac{\partial g}{\partial s}| \cdot L$$

$$\to 0$$

as $t \to \infty$. □

Lemma 15. *The L^2 norm of $\frac{\partial^3 g}{\partial s^3}$ converges to zero as $t \to \infty$.*

Proof. We compute

$$\frac{d}{dt}\int(\frac{\partial^3 g}{\partial s^3})^2 ds$$

$$= \int -\frac{2}{g^2}(1-(\frac{\partial g}{\partial s})^2)(\frac{\partial^4 g}{\partial s^4})^2 + \frac{24}{g^4}(\frac{\partial g}{\partial s})^2(1-(\frac{\partial g}{\partial s})^2)(\frac{\partial^3 g}{\partial s^3})^2$$

$$-\frac{44}{g^3}(\frac{3}{11}-(\frac{\partial g}{\partial s})^2)\frac{\partial^2 g}{\partial s^2}(\frac{\partial^3 g}{\partial s^3})^2 + \frac{1}{g^2}(\frac{\partial^2 g}{\partial s^2})^2(\frac{\partial^3 g}{\partial s^3})^2$$

$$+\frac{8}{g^2}\frac{\partial g}{\partial s}\frac{\partial^2 g}{\partial s^2}\frac{\partial^3 g}{\partial s^3}\frac{\partial^4 g}{\partial s^4} - \frac{120}{g^6}(\frac{\partial g}{\partial s})^4(1-(\frac{\partial g}{\partial s})^2)(\frac{\partial^2 g}{\partial s^2})^2$$

$$+\frac{248}{g^5}(\frac{\partial g}{\partial s})^2(\frac{15}{31}-(\frac{\partial g}{\partial s})^2)(\frac{\partial^2 g}{\partial s^2})^3 - \frac{96}{g^4}(\frac{1}{8}-(\frac{\partial g}{\partial s})^2)(\frac{\partial^2 g}{\partial s^2})^4$$

$$+\frac{32}{g^3}\frac{\partial g}{\partial s}(\frac{\partial^2 g}{\partial s^2})^3\frac{\partial^3 g}{\partial s^3}.$$

When t is sufficiently large, we have

$$\frac{d}{dt}\int(\frac{\partial^3 g}{\partial s^3})^2 ds \le -C_1\int(\frac{\partial^4 g}{\partial s^4})^2 ds + \sup(\frac{\partial^3 g}{\partial s^3})^2 + \int(\frac{\partial^2 g}{\partial s^2})^2 ds$$

for some $C_1 > 0$.

But

$$\sup(\frac{\partial^3 g}{\partial s^3})^2 \le \int|\frac{\partial}{\partial s}(\frac{\partial^3 g}{\partial s^3})^2|ds$$

$$= 2\int|\frac{\partial^3 g}{\partial s^3}|\cdot|\frac{\partial^4 g}{\partial s^4}|ds$$

$$\le \frac{C_1}{2}\int(\frac{\partial^4 g}{\partial s^4})^2 ds + \frac{2}{C_1}\int(\frac{\partial^3 g}{\partial s^3})^2 ds,$$

then

$$\frac{d}{dt}\int(\frac{\partial^3 g}{\partial s^3})^2 ds \le -\frac{C_1}{2}\int(\frac{\partial^4 g}{\partial s^4})^2 ds + \frac{2}{C_1}\int(\frac{\partial^3 g}{\partial s^3})^2 ds + \int(\frac{\partial^2 g}{\partial s^2})^2 ds.$$

Note that

$$\int(\frac{\partial^3 g}{\partial s^3})^2 ds = \int -\frac{\partial^2 g}{\partial s^2}\frac{\partial^4 g}{\partial s^4}ds \le (\int(\frac{\partial^2 g}{\partial s^2})^2 ds \cdot \int(\frac{\partial^4 g}{\partial s^4})^2 ds)^{\frac{1}{2}}.$$

So if we assume that

$$\int(\frac{\partial^3 g}{\partial s^3})^2 ds > C_0\int(\frac{\partial^2 g}{\partial s^2})^2 ds$$

for some $C_0 > 0$ to be determined, then we get

$$\int(\frac{\partial^3 g}{\partial s^3})^2 ds \le C_0^{-1}\int(\frac{\partial^4 g}{\partial s^4})^2 ds.$$

Hence

$$\frac{d}{dt}\int(\frac{\partial^3 g}{\partial s^3})^2 ds \le -(\frac{C_0 C_1}{2}-\frac{2}{C_1}-\frac{1}{C_0})\int(\frac{\partial^3 g}{\partial s^3})^2 ds.$$

If we take C_0 large enough, for instance,

$$C_0 > \frac{1 + \frac{2}{C_1} + \sqrt{(1 + \frac{2}{C_1})^2 + 2C_1}}{C_1},$$

then

$$\frac{d}{dt} \int (\frac{\partial^3 g}{\partial s^3})^2 ds \leq - \int (\frac{\partial^3 g}{\partial s^3})^2 ds.$$

Therefore, either $\int (\frac{\partial^3 g}{\partial s^3})^2 ds$ decays exponentially, or it is comparable to $\int (\frac{\partial^3 g}{\partial s^3})^2 ds$. In either event, it decreases to zero. $\qquad\square$

Since

$$\sup(\frac{\partial^2 g}{\partial s^2})^2 \leq \int |\frac{\partial}{\partial s}(\frac{\partial^2 g}{\partial s^2})^2| ds$$
$$= 2 \int |\frac{\partial^2 g}{\partial s^2}| \cdot |\frac{\partial^3 g}{\partial s^3}| ds$$
$$\leq 2(\int (\frac{\partial^2 g}{\partial s^2})^2 ds)^{\frac{1}{2}} \cdot (\int (\frac{\partial^3 g}{\partial s^3})^2 ds)^{\frac{1}{2}}$$
$$\to 0$$

as $t \to \infty$, then we can see that

$$K(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = K(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) = -\frac{1}{g}\frac{\partial^2 g}{\partial s^2} \to 0,$$

and

$$K(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = \frac{1}{g^2}(1 - (\frac{\partial g}{\partial s})^2) \to \alpha.$$

For the scalar curvature we have

$$R(t) \to 2\alpha.$$

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